J-003-1161001-N Seat No. ______

M. Sc. (Sem. I) Examination

January - 2020

Mathematics : CMT-1001

(Algebra - I)

Faculty Code : 003
Subject Code : 1161001-N

Time : $2\frac{1}{2}$ Hours] [Total Marks : **70**

Instructions : (1) All questions are compulsory.
(2) Each question carries 14 marks.

1 Answer any seven short questions : **7×2=14**

(i) Define order of an element g in a group G. In standard notation $A_3 = \left\{ \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \right\}$, write down order of element $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$ in A_3.

(ii) For a group G, prove that G has only one identity in it.

(iii) Prove or disprove, the set of all units of $\mathbb{Z}_{10}$ $\left[U\left(\mathbb{Z}_{10}\right)\right]$ is a cyclic group under multiplication.

(iv) Write down two generators of the group $\left(\mathbb{Z}_{10}, +_{10}\right)$.

(v) Define center of a group G i.e. $Z(G)$.

(vi) Give definition of a cyclic group. Write down two generators of the cyclic group $\left(\mathbb{Z}, +\right)$.

(vii) Define terms : Isomorphism of groups and automorphism.

(viii) Give statement of Sylow's first theorem.

(ix) Give an example of a non-commutative ring.

(x) Define terms : Zero divisor and Integral domain.

2 Attempt any **two** : **2×7=14**

(a) In standard notation prove that $Z(G)$ is a normal subgroup of G, where G is a group.

(b) Let G be a group and H, K be two subgroups of G. Let $HK = \{hk \mid h \in H, k \in K\}$. Prove that HK is a subgroup of G if and only if $HK = KH$.

(c) Let H be a non-empty finite subset of a group G and $ab \in H, \ \forall a, b \in H$. Prove that H is a subgroup of G.

3 Attempt following both : **2×7=14**

(a) Let G be an abelian group and $H = \{x \in G \mid x^2 = e\}$, $K = \{x^2 \mid x \in G\}$. Prove that H and K both are subgroup of G.

(b) Prove that every finite group G is isomorphic to a permutation group (a subgroup of S_G).

OR

3 Attempt following both : **2×7=14**

(a) Give definition of a field. Give an example of a field. Also prove that a finite Integral domain is always a field.

(b) Prove that $\mathbb{Z}_p$ is a field, where p is a prime. Give definition of an ideal of a ring R. Write down number of ideals of $\mathbb{Z}_p$.

4 Attempt any **two** : **2×7=14**

(a) Let G and G' be two groups and $\phi : G \to G'$ be a group homomorphism. In standard notation prove that, $G/_{Ker\,\phi} \simeq I_m(\phi)$ as subgroup of G', where $I_m(\phi) = \{\phi(g) \mid g \in G\}$.

(b) Let G be a group and $I_n(G) = \{i_g : G \to G$ defined by $i_g(x) = g\,x\,g^{-1}, \forall\, x \in G / g \in G\}$. Prove that i_g is an automorphism on G. Also prove that $I_n(G)$ is a subgroup of $Aut(G)$, where $Aut(G)$ is the set of all automorphisms of G.

(c) (i) Let G be a group and $g^2 = e, \forall\, g \in G$. Prove that G is an abelian group. **3**

(ii) Let G be an abelian group. Prove that $(ab)^n = a^n b^n$, $\forall\; a, b \in G$ and $\forall\; n \in \mathbb{N}$. **4**

5 Attempt any **two** : **2×7=14**

(a) Let H, N be two subgroups of a group G and N is a normal subgroup of G. In standard notation prove that $\frac{HN}{N} \simeq \frac{H}{H \cap N}$ as group.

(b) Let H, K be two Sylow p-subgroups of a finite group G. We define a relation "$\sim$" on G as follows :

For any $g_1, g_2 \in G$, we say $g_1 \sim g_2$ if $g_2 = a\,g_1\,b$, for some $a \in H$ and $b \in K$. Prove that "$\sim$" is an equivalence relation on G.

(c) Let R be a commutative ring with unity and M be an ideal of R. Prove that M is maximal ideal of $R \Leftrightarrow R/M$ is a field.

(d) State and prove third isomorphism theorem. (Third Fundamental Theorem) of group theory.